Exercise 20

Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

Solution

Use the definition of the derivative to prove the result.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(\cos x) \\
&= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) \\
&= \cos x \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) - \sin x \lim_{h\to 0} \left(\frac{\sin h}{h} \right) \\
&\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \cos x \lim_{h\to 0} \left[\frac{\frac{d}{dh}(\cos h - 1)}{\frac{d}{dh}(h)} \right] - \sin x \lim_{h\to 0} \left[\frac{\frac{d}{dh}(\sin h)}{\frac{d}{dh}(h)} \right] \\
&= \cos x \lim_{h\to 0} \left[\frac{(-\sin h)}{1} \right] - \sin x \lim_{h\to 0} \left[\frac{(\cos h)}{1} \right] \\
&= (\cos x)(0) - (\sin x)(1) \\
&= -\sin x
\end{aligned}$$

Note that $(\cos h - 1)/h$ and $\sin h/h$ are $0/0$ indeterminate forms when $h \to 0$, so l'Hôpital's rule can be applied to evaluate the limits.